

## Wavelets and Their Applications

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In 1909, the word "wavelet" first appeared in a thesis by Alfred Haar. Jean Morlet and the team at the Theoretical Physics Center working under Alex Grossmann in France first proposed the modern concept of wavelets, and the methods of wavelet analysis have been developed mainly by Y. Meyer and his colleagues. Wavelets are families of mathematical functions that decompose data into different frequency components. Derived from multi-resolution analysis, wavelets enable us to analyze data according to scale.

Fourier analysis also decomposes data into different frequencies. It uses periodic functions such as sine and cosine. It tells us what frequency our data have, but it can't tell us where a particular frequency occurs. Wavelets are the shifted and dilated versions of one single wavelet function. Using wavelets, we are able to localize a particular frequency.

We shall study wavelets from multi-resolution analysis. The function space considered here is  $L^2(\mathbb{R})$ . Consider a sequence  $\{V_i\}_{i \in \mathbb{Z}}$  of closed subspaces of  $L^2(\mathbb{R})$  satisfying the following conditions:

$$1 \quad V_i \subset V_{i+1}$$

$$2 \quad \overline{\bigcup_{i \in \mathbb{Z}} V_i} = L^2(\mathbb{R})$$

$$3 \quad \bigcap_{i \in \mathbb{Z}} V_i = \{0\}$$

$$4 \quad f(x) \in V_i \Leftrightarrow f(2x) \in V_{i+1}$$

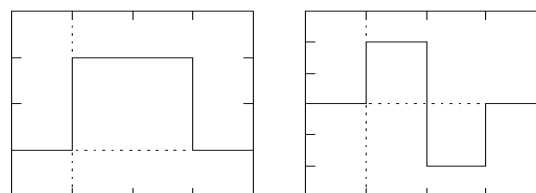
$$5 \quad f(x) \in V_0 \Leftrightarrow f(x+1) \in V_0$$

$$6 \quad \exists \phi \in V_0 \text{ such that } \int_{-\infty}^{\infty} \phi(x) dx \neq 0 \text{ and } \{\phi(x-j)\}_{j \in \mathbb{Z}} \text{ is a Riesz basis of } V_0$$

The structure of  $V_0$  determines the structure of all subsequent  $V_i$ , and the scaling function  $\phi_i$  generates a basis for each of the subspaces. Each time we increase the index of the subspace by

1, we increase the resolution by a factor of 2, which means more detail is shown in subspace  $V_{i+1}$  than  $V_i$ . Therefore, for each  $V_i$ , there exists an detail subspace  $W_i$  such that  $V_{i+1} = V_i \oplus W_i$ . Any function  $f \in L^2(\mathbb{R})$  can be represented by  $f = \sum_{i \in \mathbb{Z}} w_i$ , where  $w_i \in W_i$ . The most important result from multi-resolution analysis is that there exists a function  $\psi \in W_0$  which generates an orthogonal basis for each subspace  $W_i$ , where  $\int_{-\infty}^{\infty} \psi(x) dx = 0$ . We call  $\psi$  the wavelet generating function.

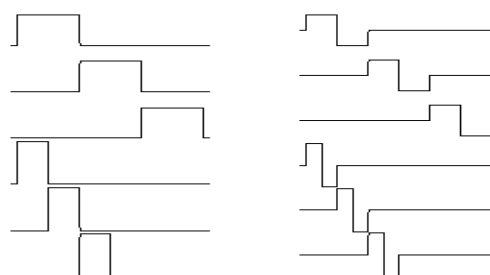
Following plots are the Haar scale function, selected basis functions, the Haar wavelet generating function and selected Haar wavelets.



(a)

(b)

(a) shows the Haar scaling function, and (b) shows the Haar wavelet generating function.



(a)

(b)

(a) shows selected basis functions of  $V_1$  and  $V_2$ , and (b) selected wavelets in  $W_1$  and  $W_2$

Wavelet analysis is widely used in signal processing. Examples of its applications are shown below.

First consider a function  $f$  plotted below.

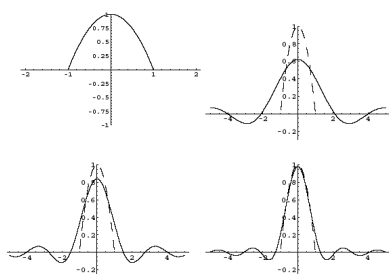
# Wavelets and Their Applications

We apply both Fourier decomposition and Haar Wavelets decomposition to approximate the function. We evaluate the approximations by looking at both maximum error and average error which are defined by following formulas:

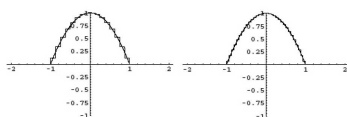
$$\text{maximum error} = \frac{\text{maximum difference}}{\text{average value of function}} \cdot 100\%,$$

and

$$\text{average error} = \frac{\text{average value of difference}}{\text{average value of function}} \cdot 100\%.$$



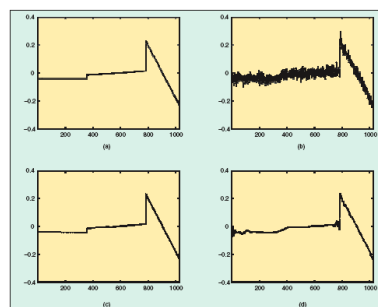
The upper-left corner shows the original function, the upper-right corner shows the Fourier approximation with 50 nonzero terms, the lower-left corner shows the Fourier approximation with 75 nonzero terms and the lower-right corner shows the Fourier approximation with 100 nonzero terms.



From left to right are the wavelet approximations with 16 and 32 nonzero terms respectively.

Fourier approximations with 50, 75, and 100 terms have maximum errors 486.31%, 389.95%, and 240.75% respectively, and have average errors 98.84%, 64.60%, and 34.12% respectively. Wavelet approximations with 16 and 32 terms have maximum errors 18.08% and 9.18% respectively, and have average errors 2.37% and 1.19% respectively. It is obvious that wavelets approximate the function value much better than Fourier with much less nonzero terms.

One of all other applications of wavelets is signal denoising. We can think of noise in a signal as fluctuations with high frequency but a low amplitude. To denoise a signal received, we can decompose the signal into different frequency bands, and set thresholds to the high frequencies. When the amplitude for a certain frequency is greater than the threshold, we pass it, otherwise we block the frequency or scale it according to different thresholding methods. Following is an example of denoising using different thresholding technique. We can see that the denoised signals preserve the singularities in the original signal. That is due to the localization of wavelet method.



(a) shows the original signal, (b) shows the noisy signal, (c) shows the denoised signal using dynamic thresholding method, and d shows the denoised signal using fixed thresholding method.

## References

Brendt Wohlberg, *Introduction to Wavelets*  
 Nicholas G. Roland, *Fourier and Wavelet Representations of Functions*  
 Martin Vetterli, *Wavelets, Approximations, and Compression*

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